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Entropy: Form Follows Function

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Abstract

Many definitions of entropy are discussed in intimacy with Boltzmann-Gibbs exponential distribution and measure theory. Tsallis introduced a definition that collapses to other definitions when parameter ‘q’ reaches unity limit. In this definition, Levy statistics are said to be used instead of Boltzmann distribution. Working with the hypothesis *“are we not wrong in being obsessed with the form of entropy functional?”* aim was to assess if Tsallis definition can be an alternate basis for Shannon’s information entropy. A mix of analogy, comparison, both inductive and deductive reasoning were employed as research enquiry tools. The process of enquiry touched many fields to which entropy is related, including quantum information theory. The enquiry concluded that entropy additivity is adopted axiomatically in most fields for good reasons, mainly emanating from measure theory, conforming to the view that the entropy has more to do with topology and volume growth rather than statistics and distributions.

Keywords: Entropy, Tsallis, Levy and Boltzmann-Gibbs distributions, topology, volume growth, additivity, quantum information theory, information theory

Introduction

A debate rages amongst the scientific community, and two distinct camps argue the functional forms of entropy they support. Entropies have been used in thermodynamics, information theory and many other fields. Kolmogorv introduced entropy of a measure preserving transformation on probability space; viz. measure theoretic entropy. In dynamic systems entropy is taken to measure rate of divergence of function orbits. In topology, volume growth, invariant measure theoretic and topological entropies are defined Gerhard Knieper. (1995). The last decade has been replete with published works that relate Tsallis entropy to every conceivable domain of science, Information Science complexity theory included.

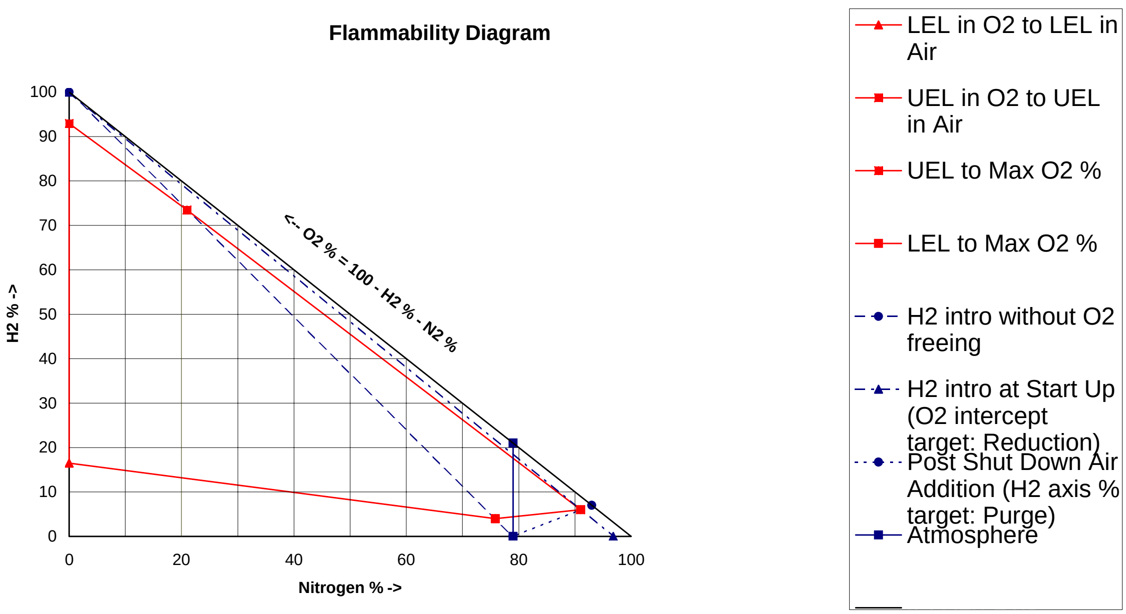
Relevance

All debates that relate to entropy allow us to explore solution mechanisms adopted using the so-called entropy maximization concept. Many of these relate to computer science and its quest for problem solving techniques. For example phase transition is observed in a 3-SAT problem (Ke Xu & Wei Li (2000). If phase transition is understood well, then the methods evolved in the process could be used for speedily solving 3-SAT like problems and we have P=NP. The interactions are very much a part of growing literature on Multi-agent systems and are an evolving area as in works of Liau, C. J. (2002). Particles in a box model of statistical mechanics has been applied to evidence, belief by Saul L. K., Jaakkola T., & Jordan M. I. (1996), to product distribution by Wolpert, D. (2003), in cooperative and non cooperative game theories, bounded rationality by Wolpert, D. (2004), for multi criteria decision making and control of multi agent systems by Lee, C. F. & Wolpert, D. (2004). Tsallis entropy has been purportedly used in scaling phenomena in Sociology, Language, Biology by West, B. J. (2004); in physics, chemistry, economics, medicine, geophysics, computer science by Gell-Mann, M. & Tsallis (Eds). (2004), by Abe S. & Okamoto Y. (Eds). (2001), and by Yamano, T. (2004); to Internet Seismic sciences by Abe S. & Suzuki N. (2003), in Chaos by Tsallis C., Plastino, & Zhang. (1997), Tsallis, Rapisarda, Baranger, & Latora. (2000). and Tsallis, Baldovin, & Schulze. (2003); in Quantum information theory by Abe S. (2004 December), and by Tsallis C., Lambert P. W., & Prato D. (2001); seemingly in everything under the sun. We notice a few have started to adopt Tsallis functional form and thought in Information theory (Kojadinovic, I., Marichal, J. L., & Roubens, M. (2005)).

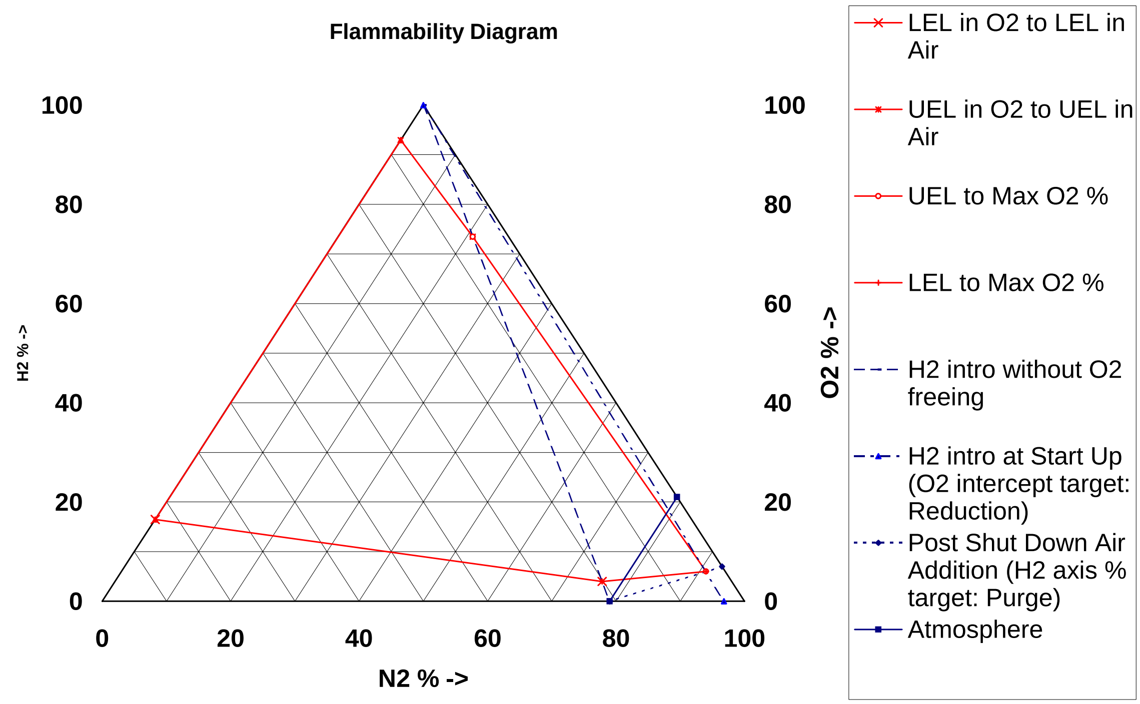
Invariance and Scaling

By covering some aspects of scale variations by some very simple examples, formal aim below is to search an answer to the question: Can we relate the concept of entropy to scales and invariance? Change is the nature of things. What does not change when a reference point of view is changed is an invariant measure. Shannon used logarithm as a measure of communicable information. Usual first example of scale issues in linear systems is scaled partial pivoting in Gaussian elimination. Use of relative coefficient magnitudes is important to preclude overflow. An interesting example applying the concept of invariance is that of rotation of axes. Chemists plot triangle diagrams to show flammability envelopes. These diagrams can be easily plotted as right angle triangle diagrams in excel. Commercial triangle graph paper is equilateral triangular. The right triangular Graph 1 can be rotated to obtain an equilateral triangular Graph 2

Graph 1



Graph 2



Solution of quartic is an example that comes from theory of invariance group relationships between coefficients of polynomials that evolved while Galois proved that polynomials with non prime compositional indices could not be solved. Feedback control systems that translate initiator signals from units of current to disparately different indicated units like those of temperature use "scaling over the signal band width" to change figures between completely differing signal carrier channel units. What is not apparent to casual observation is that unless additivity holds in each carrier domain, this translation is impossible. Phenomenological scaling is used to establish links and analogy between phenomena that differ in scales using invariants. Several such phenomenological models follow non BG distributions and Levy statistics.

Phase Transition

Supporters of Tsallis, C. (1988) entropy originally suggested that his form enables describing phase transitions and characterize them by a parameter q. Phase transitions were being used to characterize hardness of problems. Thus Tsallis q parameter came to be known as a measure of complexity; Yamano, T. (2004). According to Occam's razor (http://en.wikipedia.org/wiki/Occam%27s_razor), when alternate explanations are available to describe a phenomenon, the simpler one that explains all facets involved is to be accepted. Since Occam's razor is a non-mathematical statement; it is untenable to accept it as the justification of Tsallis, C. (1988) entropy as the claimant that describes phase transitions. Phase transitions have topological origin as investigated by Angelani et al (2004). Thus for a statistical mechanics model without non-homogenous interactions, it could be unreasonable to accept that Tsallis, C. (1988) entropy contributes anything of substance over Shannon-Gibbs-Boltzman formulation. In the same vein Tsallis 'q' parameter interpreted as a measure of complexity by Yamano, T. (2004) becomes questionable. However Occam's razor; itself is without a proof.

Topology and Projective Geometry

Kiehn, R. M. (2004) has detailed explanations of non-equilibrium stationary states and entropy based on thermodynamics, topology and geometry. Thermodynamics is a topological science. This is clear from a few examples below.

(1) One of the oldest results in topology is that number of vertices and two dimensional faces of a bounded convex polyhedron exceeds the number of edges by two (Dey, G. & Edelsbrunner, n.d.). Compare this to the famous Gibb's phase rule that is often introduced with the memory aid as below:

Police Force is Chief plus two, where P stands for phases, F stands for degrees of freedom, and C stands for components.

(2) Thermodynamic varieties such as Free energies and Entropies are independent of path; they depend only on the end states. In Cartan's topology and projective geometry, the path goes around the infinity and returns to another point. Thus the absence of the concept of in-betweenness is shared by Thermodynamics and projective geometry and topology.

Entropy is an extensive variable. Additivity of extensive variables is a property of projective transformations: Kiehn, R. M. (2004). Projective geometry is at the upper most level of geometric generalizations. It has its invariants in the six cross ratios with one functionally independent ratio k (Show Table of cross ratios and graph). Let us formally define a measure to grasp what additivity of a projective transformation is: " *A measure on a topological space X is just any function m from the family of subsets of space X to non-negative real numbers satisfying the property of countable additivity*": Amenta N., Bern M., Eppstein D., & Teng S. H. (2000), Projective transformations are countably additive. This situation happens to fit perfectly to Boltzmann-Planck's choice of logarithm and unfortunately, does not fit at all to Tsallis's choice of a non-additive deformed logarithm. It is clear that in above arguments we first called entropy an extensive variable and then invoked additivity of extensive variables in projective geometry. This is *a circular logic and hence in reality, entropy is extensive (additive) by axiom. The axiomatization of countable additivity was the fundamental contribution of Kolmogorov's measure theory.*

Novelty, Generalization and Innovation

In Appendix 1, a somewhat provocative extension of Tsallis generalization defines an entire and novel new branch of mathematics called "q deformed mathematics" including a "q deformed projective geometry". The purpose of this novelty is to demonstrate the connotations of rejecting additivity in entropy definition. It shows how easy it is to innovate and generalize to get a q deformed world. For example a chemist could obtain q deformed pH of acids and bases using the q deformed logarithm. Without doing any experiment, we can confidently state that no physical experiment will validate the q deformed pH measure. We accept countable additivity in the real world and reject the novelty of all generalizations and innovations like Appendix 1.

However, whatever happens to the complex world? In fine print that we did not state or reproduce, Angelani et al (2004) use a complex number in the partition function exponential to derive phase transition and conventional entropy. However, they forget to address the imaginary part and it is left dangling in free or phase space, after they had no use remaining for it.

Is it possible that we have states in the partition function that are sums of logical states with complex coefficients?

Can we then define a countable additivity on real and imaginary parts separately? Or at least map the imaginary part on to real domain and continue using something akin to conventional countably additive measure?

Can we use both the available probabilities in the projective geometry, but use one for the real variable and the other for the imaginary variable? Of course we just innovated another projective geometry in complex space, and have to radically alter Kolmogorov's concept of measure telling us about mappings to positive real numbers to address complex numbers as well. The answer to first of the above questions is yes. This space is used to define the Qubit in Quantum Computing. The remaining two questions are also answered yes (may be in a modified form), that must logically flow from the first yes. All this brings us to a rather hard to answer question, is Tsallis entropy then suited for the job in the Quantum arena? Unfortunately in the above two answers we already implicitly answered this in the negative. We shall expand on this topic a bit, later in this work under the heading quantum information theory.

Measure Theory

Measure theory is at the foundation of information science and perhaps most of the science. This fact is usually not apparent, due to years of schooling in determinism. Probability involves random occurrence of events. Two successive events can be independent or dependent on each other or be a sequence of an exhaustive search process. The dependent events could further be mutually exclusive or second could relate to the prior one in some other way. Kolmogorov re-enunciated the laws of probability on a geometric foundation, formally called measure theory. These laws and current challengers that bring additivity into question are listed below:

The total compound probability of instantiations of events A, B and C of the possible states of the world is shown in Table 1.

Table 1. The laws of probability

Event type	The total probability law
Any two events	P(AUB)=P(A)+A(B)-P(A∩B)
Disjoint events	As above, but P(A∩B)=0.0, Countable additivity
Dependent events	As above, but P(A∩B)=P(A)P(B|A)=P(B)P(A|B), dependence may relate to non additivity
Three events	P(AUBUC)=P(A)+A(B)+P(C)-P(A∩B)-P(B∩C)-P(A∩C)+P(A∩B∩C)

If and only if either P(A) or P(B) is zero then the events are said to be mutually exclusive and complimentary and the compound probability is additive (http://mathworld.wolfram.com/CountableAdditivityProbabilityAxiom.html). If it is not the case and P(A,B) exists and the compound probability is non-additive. In terms of logic symbolism

{{Pa==0} ˅ {Pb==0}} ↔ {P is additive}

¬{{Pa==0} ˅ {Pb==0}} ↔ {P is not additive}

P(A∩B)=P(A)P(B) if events are independent

When additivity is allowed and used, the belief is evidence. When combination is allowed, belief is a generalized probability, Halpern & Fagin. (1992). Halpern and Fagin apparently conclude to leave the decision as to which one to use on domain specialists. Information entropy was derived in thermodynamics under the assumptions disjoint events, factorizability and a not well-clarified " homogeneity " assumption. These assumptions about events make the compound probability used in the occurrence of events in thermodynamics additive. Dependent events mean a term that caters to any interaction between two events must be taken into consideration. Central theme in the issue of entropy functional form is the assumption of additive probabilities and independence of events. It can be easily misconstrued from the Tsallis form that the definition of compound probability is what Tsallis followed and his work caters to interactions, but that is not so.

Fuchs, C. A. (1995) gives very detailed derivations of the following on Information theoretic and probabilistic grounds in his thesis:

Error probability quantum and classical;
Fidelity, quantum and classical;
Renyi overlaps, quantum and classical;
Kullback Information, quantum and classical;
Accessible Information, quantum and classical;
Shannon's entropy.

It becomes clear from derivations reproduced by Fuchs, C. A. (1995), Appendix 2 that his so called q factor tending to unity needed to develop Shannon's entropy asymptotically from probability theory has inspired the Tsallis q factor. However, the effort in the information theory derivations is about finding a true distribution from a host of false distributions and only one of them is true. That happens to be the one when q factor is at its unity limit. And then it is Shannon-Boltzmann-Gibbs distribution. In all of Fuchs, C. A. (1995) set of probability based information theory derivations, the issue of compound probability is not approached with homogeneity & heterogeneity or interactions in mind. Thus it is clear that Tsallis entropy definition emanates from an incomplete derivation of Shannon's entropy embedded in information theory. And Tsallis definition, even according to information theory can be true only in the unity limit of the q factor; there it collapses to Shannon's definition. Thus we conclude that in fact we have no "new" definition of entropy that emanates from Information theory.

Quantum information theorists have advocated two further approaches to probability theory due to its internal debates.

Khrennikov. A. (2005) calls any two quantum measurements as those of vectors. He then uses law of vector addition to develop a probability law as below (bold face is a vector):

$\mathbf{C}=\mathbf{A}+\mathbf{B}$;

$\mathbf{C}.\mathbf{C}=(\mathbf{A}+\mathbf{B}).(\mathbf{A}+\mathbf{B})$

$|C^2|=|A^2|+|B^2|-2|ABCos(\theta)|$

The last form has been analogously compared to Born's rule and is used to define a probability law as: $P(C)=P(A)+P(B)-P(A\cap B)$

The cosine term is normalized in some way. When angle between vectors changes quadrants, the sign of cos term goes positive and use of Euler's exponential-trignometric relation results in hyperbolic cosine in place of cos term. Andrei calls this "hyperbolic representation" of probabilities and calls all of this work as "contextual probability", i.e. a euphemism for compound conditional probability. However such a contrived description is not justified. Remember that the entropy and probabilities arrived on the scene due to interaction issues and while minimizing thermodynamic free energy. The interaction term can have either a positive or negative sign, based on energy balance of the system. Hyperbolic behavior does indeed exist in physically real non-homogeneous systems. We will visit this by giving physical examples in liquid mixture at a later stage.

Salgado, R. (2002) evolve a new line of thought. Citing failure of additivity for the famous double slit experiment, they propose a generalization of classical probability measure and state that a quantum probability is not additive at two event level and define an interaction non zero term I2 in the compound summand of event probabilities. However from three events (each taken as set having sum of the square norms of additive quantities) onwards, they state that quantum mechanics follows normal classical probability law. This has an analogy throwback to Gleason's theorem (Caves C., C. Fuchs C., Manne K., Rennes J. (2003) that states quantum measurements for dimensions greater than two are feasible, notwithstanding Hiesenberg's uncertainty principle. But beyond that, Salgado, Sorkin generalization of measure appears to be artificial and contrived as well.

Interactions and Theories of Liquid's

When interactions and heterogeneity are involved there is one true "mixture" distribution that includes interaction between more than one dis-similar entities participating in the chance. And there are false "mixture distributions".

Within the true mixture distribution itself, compound probability must apply and yet the overall probability has to be bounded by unity. When mixtures shift to pure states, the probabilities should remain bounded by unity. Since this starts to get confusing, it is time to turn to an area of science that has had the debates about this ongoing for years. Such discussions are about Entropy of mixing in liquid's theory. We of course begin with a clear understanding that science does not yet understand entropy of mixing very well and we are all still learning.

Liquids, solids, social teamwork, extreme programming pairing, HCI, biological systems, macro-economy's are physically the easiest examples of interactions. We shall address quantum entanglements as an example at a later stage. This section draws practitioner's ideas into the world of Information theorists.

It is known for long that theories of liquids and solids are far from perfect. However, engineers have had to deal with predictions for liquid systems, even when as we know, basic thermodynamics handled interactions and phase transitions unsatisfactorily. Applied literature on liquids is far ahead of Information theorist's literature. Engineers skirted the entropy debate completely and moved ahead with their work. Let us see how. Liquids theories of applied domain do not introduce the interactions by modifying or using Shannon's entropy definition, but by separately adding functional terms to the upper level, the system configuration free energy. The entropy as gets derived from *free energy minimization* (*the principle of least effort in team works*) is simply, accepted.

The interactions can be basically grouped into two categories, those that are homogenous and those that are not. An excellent example of interactions in biological social systems is a flock of birds flying in an arrowhead formation. It is now common knowledge that if the birds fly in an arrowhead, the effect of flapping wings of neighbor reduces energy required to fly and the flock can take longer flights than individual birds due to this energy minimization. (See Figure 1.)

Figure 1. A flock of geese flying in an arrowhead V
Retrieved from http://www.loc.gov/rr/scitech/mysteries/geese.html

What Information theorists call as "entropy maximization" is in reality, not entropy maximization but "energy minimization". Social scientists frequently quote this example in collaborative leadership and as a justification for encouraging teamwork and it is perhaps the model example for collaboration and cooperation in a Multi agent system. The homogenous interactions arise from topology of a problem and are related to entropy and information loss. Topology and entropy are intimately linked as shown by Angelani et al (2004). Homogenous interactions can also be multi-entity interactions and can be categorized again on the basis of number of entities involved in the interaction. Probability theory incorporates many body interactions as well. For example:

$P(AUBUC)=P(A)+A(B)+P(C)-P(A\cap B)-P(B\cap C)-P(A\cap C)+P(A\cap B\cap C)$; if events are pair wise mutually exclusive, then ternary interaction terms vanish. The transition order and number entities involved delink themselves for first and second order transitions, however for greater than second order transition the order and number of entities is linked by Euler characteristics as shown by Percus, A. G. & Martin, O. (1998).

The non-homogenous interactions are the ones that were introduced as application example in case of liquids, solids, social team work, terrorism etc. at the beginning. Heterogeneous interactions can be further classified into three categories viz. attractive, neutral and repulsive. The involvement of only two agencies in interactions makes them binary interactions. If the interaction between two is effected via a third party, then it is a ternary interaction. This is akin to " k nearest neighbour " ideas of Percus, A. G. & Martin, O. (1998). Heterogeneous interactions modify system models like the statistical mechanics model. However the manner in which they get introduced into the entropy of the system is very different than homogenous interactions. Though such interactions also operate with and through the system topology, they are present owing to heterogeneity's own inherent nature and behaviour and specifically not only due to mathematics of the topological space. Heterogeneous interactions must be configured and added externally to the system model function and computed and checked separately for their information loss contribution. The overall picture of interactions is shown in Figure 2.

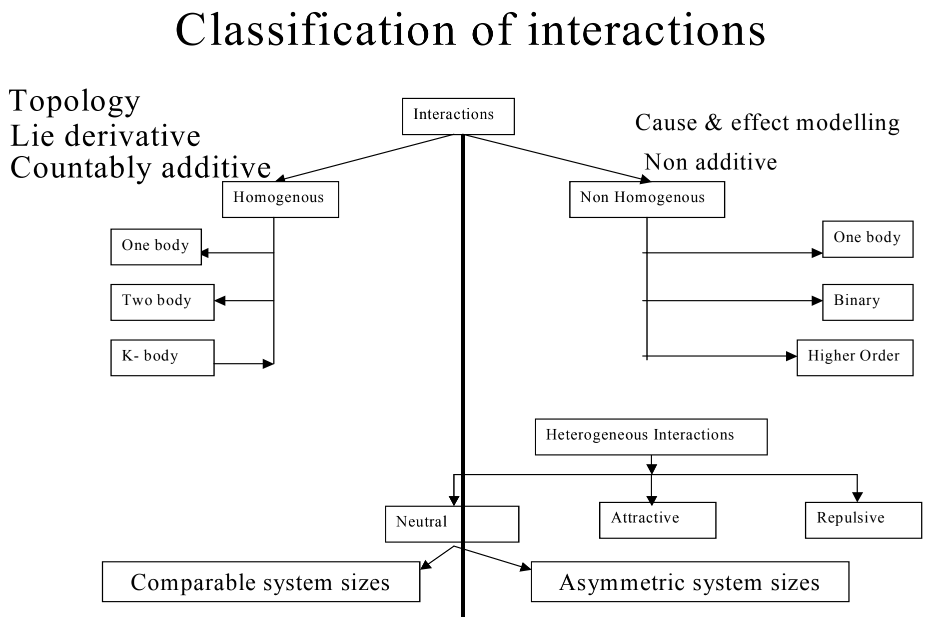


Figure 2. Interactions

In agent systems, when multi-goal programming is desired, these interactions can be introduced as normalized objectives that approach a goal. Two useful points from the theories of liquids are firstly making goals and variables bounded at both ends and expressing them as scaled variables within the bounded range as shown below and secondly to normalize the contributions to the goals so that probabilities add to one, this is a way of ensuring finite additivity instead of a countable one and it ensures that proper weights are given to the interactive goals with non interactive ones.

Scaled Goal = (Unscaled goal – Lower Bound)/(Upper Bound-Lower Bound), similarly for variables.

The need for this normalization arises because pure liquid specie need to obey additivity and probabilities must lie between zero and one. However, predicted interactions may some times end up leaving a residual in the pure liquid state when purified from an initially mixed state or the other way round. This is very analogous to what quantum information theorists are now finding in experimental data on pure and mixed states and entanglement distillation. Two molecules of same liquid may have attractive or repulsive parts, bringing neighbours close or repulsing them leading to an element of non additivity in the Interactions. This is where compound probability could be used, but through energy balance; not in any other purely mathematical way.

Let us close this part with two quick examples of liquid systems (Shown in Table 2) that involve heterogeneous interactions (Gmehling & Onken. (1977) and graph their behaviour.

Table 2. The deviant liquid mixtures

Mixture	Substances involved
1	Acetone (1-1), Chloroform(1-2)
2	Acetone (2-1), Methanol (2-2)

The liquid activity coefficient is a measure of deviation of the mixture from ideal behaviour where perfect countably additive Boltzmann Gibbs entropy applies. Values greater than unity are reported for mixture 1, less than unity for mixture two. This graph (Figure 3) shows physical example of what is termed hypothetically existent hyperbolic probability law by Khrennikov. A. (2005). But the mixture specificity of the extent of hyperbolicity indicated by the data says, one could not state hyperbolic probability law was the causal set for this behaviour, but other complex system characteristics are. There are voluminous further weird (eg two liquid phases like Kerosene-water, azeotropy etc.) and very varied real physical data in the Dechema series gathered in Gmehling & Onken. (1977).

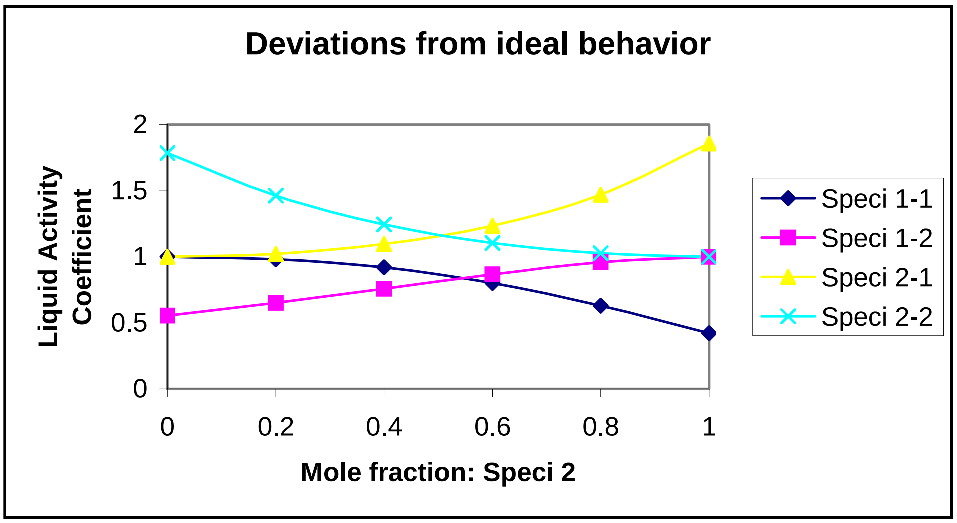


Figure 3 The plot of deviant liquid mixtures

The real world is far from ideal and above analogies we made here rather speculatively relying on Tsallis, C. & Souza A. M. C. (2003). Stability of entropy for superstatistics. *Physics Letters A 319*, 273-278.

Van der Waals (1894) theories are gradually turning out to be true in quantum domain as found by Castorina, Riccobene, & Zappala (2005). How are such observations going to be addressed within quantum information theory?

Quantum Information Theory

It is clear that for Tsallis entropy to be valid, non additivity must be valid. We will look into the evolution of additivity aspect in Quantum Information theory.

The Qubit

Quantum information is fundamentally different in character to classical computing information. Classical computing information is in bits, probabilistic information works in pbits and quantum information is in qubits. (See Figure 4.)

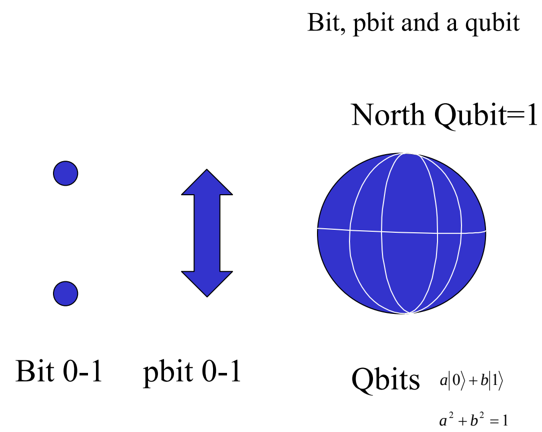


Figure 4 The bit, pbit and qubit

Visually, the qubits can be shown on a Bloch sphere, where infinitely many values between zero and one could be taken up, whereas a pbit takes only fractional values on the strip from zero to one and bit just takes two very limited values, zero and one. Due this fundamental difference in the representation of information, Shannon's proof that a communication channel has a well defined information carrying capacity and the formula he gives for it have to be modified.

Significance of Entanglement

The fields of quantum computing, teleportation and cryptography are feasible only because of what is called quantum entanglement. (See Table 3.) Keyl M. (2002) gives a vivid demonstration of this.

Table 3 Significance of entanglement

Sr No	Quantum operation	Without Entanglement	With Entanglement
1	Joint measuring device (e.g. Momentum, position): Hisenberg's principle	No	Yes, when POVM averages are used; violation of Bell inequalities,
2	Quantum cloning or copying	1 is No hence No	Yes, because 1 is yes
3	Teleportation	1 is No hence No	Yes, because 1 is yes
4	Dense coding	1 is No hence No	Yes, because 1 is yes
5	Error correction	1,2,3,4 No hence No	Yes, because 1 is yes

Measurement of State

More correct statement of Hisenberg's principle is that velocity and momentum cannot be projectively measured together. By Gleason's theorem (Caves C., C. Fuchs C., Manne K., Rennes J. (2003) and Born's rule, measurements of quantum variables are possible on Hilbert spaces of dimension greater than two. This measurement is usually made as a positive operator valued measurement (POVM). Those who utilize quantum computing usually resort to its operational vision. That is:

1 Probability is interpreted in a Bayesian manner; in a non-contextual fashion. Frequency interpretation is not used.

2 There is no attempt to insist on mixed state being convex as were the pure states;

There is no attempt made to defend use of many world's interpretation of quantum mechanics that is involved. The wave function collapse issue in this approach is handled by making the observer a part of the superposition of states. The state vector is objective and inclusion of observer keeps the whole system objective as far as probabilities are concerned. Though objective, the measurement cannot say what was measured or the outcome in each branch of the many worlds. This is an instance of the axiom of choice (http://en.wikipedia.org/wiki/Axiom_of_choice).

Distinguishability

To use the quantum information one must be able to distinguish between two different states of it. This can only be done by statistical comparison of distributions of measurement results. Thus methods as tabulated in Table 4 are needed.

Table 4 The measures of distinguishability

Measures of distinguishability	For Probability Distributions	For Quantum states
Probability of error	$P_e = \sum_{b=1}^{n} \min\{\pi_0 p_0(b), \pi_1 p_1(b)\}$	$P_e(\rho_0 \vert \rho_1) = \min_{E_b} \sum_{b=1}^{n} \min\{\pi_0 tr(\rho_0 E_b), \pi_1 tr(\rho_1 E_b)\}$
Relative information	$K(p_0 / p_1) = \sum_{b=1}^{n} p_0(b) \ln\left(\frac{p_0(b)}{p_1(b)}\right)$	$K(\rho_0 / \rho_1) = \max_{E_b} \sum_{b=1}^{n} tr(\rho_0 E_b) \ln\left(\frac{tr(\rho_0 E_b)}{tr(\rho_1 E_b)}\right)$
Renyi overlaps	$F_\alpha(p_0 / p_1) = \sum_{b=1}^{n} p_0(b)^\alpha \, p_1(b)^{1-\alpha}$	$F_\alpha(\rho_0 / \rho_1) = \min_{E_b} \sum_{b=1}^{n} (tr(\rho_0 E_b))^\alpha (tr(\rho_1 E_b))^{1-\alpha}$
Fidelity	$F_{0.5}(p_0 / p_1) = \sum_{b=1}^{n} \sqrt{p_0(b)} \sqrt{p_1(b)}$	$F_{0.5}(\rho_0 / \rho_1) = \min_{E_b} \sum_{b=1}^{n} \sqrt{tr(\rho_0 E_b)} \sqrt{tr(\rho_1 E_b)}$
Mutual information	$= H(\pi_0 p_0 + \pi_1 p_1) - \pi_0 H(p_0) - \pi_1 H(p_1)$	
Accessible information		$I(\rho_0 \vert \rho_1) = \max_{E_b} \sum_{b=1}^{n} (\pi_0 tr(\rho_0 E_b) \ln\left(\frac{tr(\rho_0 E_b)}{tr(\rho E_b)}\right) +$ $+ \pi_1 tr(\rho_1 E_b) \ln\left(\frac{tr(\rho_1 E_b)}{tr(\rho E_b)}\right))$

The accessible information is bounded above and below. The upper bound of accessible information is the Holevo bound. Accessible information is limited by Von Neumann entropy. It equals Shannon's entropy only when the qubits are orthogonal in the quantum state space; for non-orthogonal qubits, it is less than Shannon's entropy. Both Von Neumann and Shannon entropies are additive.

It is clear from Table 3 **Significance of entanglement** that entanglement is what enables quantum computing, teleportation, dense coding, error correction and cryptography. It is necessary to have a measure of entanglement to assess if entanglement in a certain system is sufficient for a given task. Qualities expected of an ideal entanglement measure are introduced through a series of axioms below.

Axioms and Additivity

To quantify entanglement is to associate a positive real number to each state of finite dimensional bipartite system. This is an axiom A0 and has obvious connection to measure defined by Amenta N., Bern M., Eppstein D., & Teng S. H. (2000) and measure as in functional analysis and that of Kolmogorov. (See Table 5)

Table 5 The axioms of quantum information theory

Axiom	Quantum Information theory	Measure theory
A1	Entanglement vanishes on separable and takes a maximum on maximally entangled states.	
A2	measure of entanglement has to be monotonic i.e. cannot increase under Local operations and classical two way communications.	Measure be monotonic
A2a	The measure be invariant under local unitaries.	Do
A3	a convex function	a convex function
A4	It is continuous	it is continuous
A5	Additive	Additive
5a	Sub additive	
5b	Weak additivity	
5c	a regularization exists for it Daubechies I. & Klauder J. R. (1985).	a regularization exists Wiener, N. (1923)

Table 6 shows status of all entanglement measures currently around in the literature vis-à-vis axioms A0-5. Y stands for valid, X for not valid and ? for unproven.

Table 6 Measures of entanglement vis-à-vis axioms

		A_0	A_1	A_2	A_3	A_4	A_5	A_{5a}	A_{5b}
E_D	Entanglement of distillation	Y	Y	Y	?	?	X	Y	Y
E_C	Entanglement cost	Y	Y	Y	?	?	?	?	Y

E_F	Entanglement of formation	Y	Y	Y	Y	Y	?	Y	?
E_R	Relative entropy of entanglement	Y	Y	Y	Y	Y	?	Y	X

It is clear that additivity A5 is not explicitly listed as a proven.

A Holevo channel can be defined using Von Neumann entropy functional. Maximizing the Holevo capacity over all probabilistic ensembles of a set of quantum states gives the information carrying capacity of that set of quantum states. When inputs entangled between different channel uses are allowed, the question of additivity arises. The additivity issues presented in various entries in Tables 4, 5 & 6 in several different forms, have been shown to be equivalent by Shor, P. W. (2003).

Equivalence of additivity questions in Quantum Information Theory

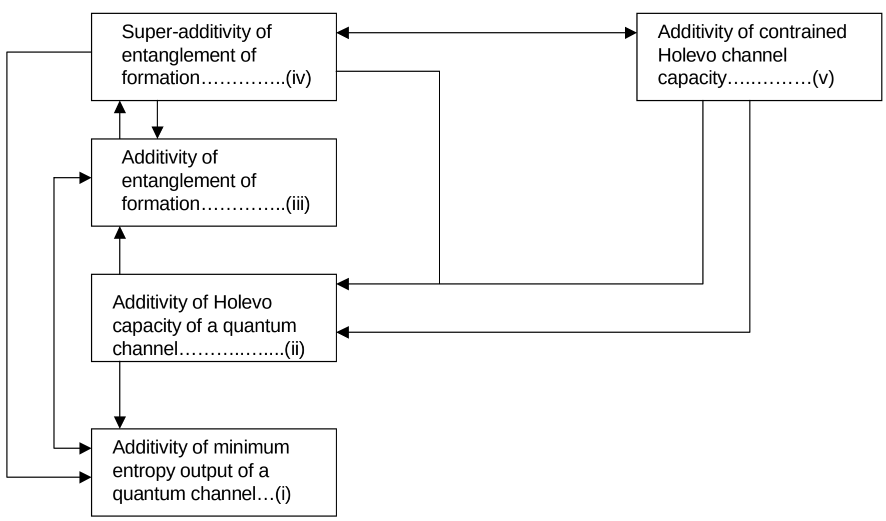


Figure 5 Tautologies of Additivity

Statement equalities (ii)=(i), (ii)=(iii), (iii)=(iv), (iv)=(iii), (i)=(iii), (iii)=(i), (iv)=(i), (iv)=(ii), (v)=(ii), (v)=(iv) and (iv)=(v) have been proven so far. If any one of the statements is true, all five are true. That makes them all tautologies. Interestingly, Table 5 shows sub-additivity of the entanglement of formation and only the entanglement of distillation to be non-additive. Other three measures are conjectured to be additive, but not proven.

Interaction and Entanglement

It is clear from liquids theory that for interacting systems, the entropy may not be additive. A normalization step that is specific to the system concerned is then needed to ensure that probabilities add up to unity. In case of quantum entanglement, the situation could be claimed to be different. But is it? Do we have extensive data on many qubit systems? The interaction here is between subsystems in the phase plane rather than between two different interacting participants. This means the entropies can potentially be thought to form from incomplete q deformation that tends to unity to eventually recover the Von Neumann entropy and obey Holevo bound.

This somehow suggests non additivity is valid in quantum domain. This has been the justification for those staking claims that Tsallis entropy is valid in quantum mechanics domain. However, as we have seen already the additivity is axiomatic. Extensive numerical search has thus far supported additivity axiom Hayashi, H., Maysumoto, K., Ruskai. M. B. et al. (2004). Perhaps the key to this is energy balance in the quantum domain. Where do the interactions between fermions and bosons fall i.e. to the right or to the left side of the interactions picture showed before?

Conclusion

No experimentation is yet documented in support of Tsallis entropy in the pure science domain. Frank, T. D. & Friedrich, R. (2004). have suggested one experiment. Interactions and their impact should be explicitly introduced into the system and analyzed. Entropy in non-homogeneous systems is not additive. Theorizations in quantum information theory are still based on few qubits. It is not clear what will be the impact of many body interactions on theories. Based on Cartan's topology, measure theory and projective geometry, for non-interacting and homogeneous reversible or irreversible continuous systems, the entropy form of Shannon-Gibbs-Boltzmann is the correct form.

In quantum information theory, the additivity issue is unresolved but Kolmogorov's measure theory is philosophically clear that non additivity means non measurable sets and any theory without additivity is simply not admissible. In function spaces unless small infinitesimal increment is "yet finitely non zero, quantized… discrete; infinities are regularized away, continuity and countable additivity hold", not all integrals that have non zero answers can be worked out.

In the non-interacting as well as interacting processes, functionally more important than the form of entropy is the concept of information loss (free energy) minimization. Form follows function.

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Appendix 1

The q deformed logarithm and its inverse function are defined as below:

$Ln_q(x)=(x^{1-q}-1)/(1-q)$; Inverse $e_q=(1+(1-q)x)^{1/(1-q)}$, when $1+(1-q)x>0$ and $e_q=0$, otherwise.

Tsallis entropy is S_q=Sum up [$p_i Ln_q(1/p_i)$]

In order to introduce the q generalization into the mean field theory, an entire branch called "q deformed mathematics" needs to be evolved on following mathematical functional lines:

The q Trignometric functions: $Cos_q\theta = \dfrac{e_q^{i\theta}+e_q^{-i\theta}}{2}$; $Sin_q\theta = \dfrac{e_q^{i\theta}-e_q^{-i\theta}}{2i}$

A q Complex number : $Z_q = re_q^{i\theta} = r(Cos_q\theta + iSin_q\theta)$

A q gamma function: $\Gamma_q = \int_0^\infty x^{-1}e_q^{n-x}dx; n>0$

A q Bessel function: $J_{pq}(x) = (\frac{x}{2})^p \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k}}{k!\Gamma_q(p+k)}$ when p is an integer.

It is clear that "q generalized projective geometry" will result from the q generalized sin functions above introduced into Pascal's Hexagram (http://www.mathpages.com/home/kmath543/kmath543.htm). These q generalizations fall back to their respective standard mathematical counterpart functions as q tends to unity.

These functions follow standard Lemma as applicable to such functions.

Appendix 2

This appendix lists the probable source of Tsallis's inspiration in evolving his formula listed in Appendix 1. Predictably, it is the derivation of the entropy based on information theory. We will not reproduce the derivation that has been reproduced in many theses. e. g. Fuchs, C. A. (1996), Boyden III, E. S. (1999). but display two of its key steps and state that this derivation is not complete unless the unity limit of q factor of Tsallis is taken.

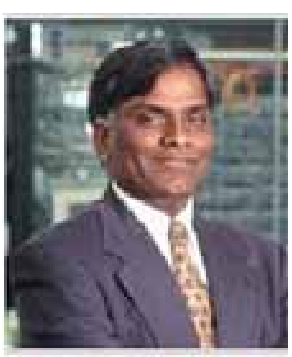
$$H(p(x) = (2^{1-q} - 1)^{-1} \left\{ \sum_{i=1}^{n} p_i^q - 1 \right\}$$

$$\underset{q->1}{Lt}[H(p(x)] = \underset{q->1}{Lt}[(2^{1-q} - 1)^{-1} \left\{ \sum_{i=1}^{n} p_i^q - 1 \right\}]$$

$$H(p) = -\sum_{x} p(x) Lg_2(p(x))$$

Then we have no new entropy other than that of Boltzmann-Gibbs-Von Neuman. Incomplete derivation, when limit is not taken cannot make a new definition. Neither is this incompleteness very Goedelian nor is it any improvement over Prigogine who talked about self-organized multiple stationary states far from equilibrium (Forman Robin's home page How many equilibria are there: An introduction to Morse Theory, available at http://math.rice.edu/~forman/.). The historical root cause of the problem is neither Herr Planck nor Herr Boltzmann justified their choice of logarithm by a derivation. A logarithm links topological volume to additivity making it a true measure. Choice of form follows the functional need, and did not need a derivation. Shannon's work is chronologically latest, prior to that is Von Neuman's quantum mechanics based definition. Boltzmann Planck's work was the earliest. The definition of entropy is about topological volume growth, not about statistics or distributions.

Biographies



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